JOURNAL OF COMBINATORIAL THEORY, Series B 29, 27–46 (1980)

Hamilton Cycles in Regular 2-Connected Graphs

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Communicated by the Editors

Received August 23, 1977

We shall prove the following result.

THEOREM 1. *Every 2-connected, k-regular graph on at most 3k vertices is hamiltonian.*

This result is best possible for $k=3$ since the Petersen graph is a non-hamiltonian, 2-connected, 3-regular graph on 10 vertices. It is essentially best possible for $k \geqslant 4$ since there exist non-hamiltonian, 2-connected, k-regular graphs on $3k+4$ vertices for k even, and $3k+5$ vertices for all k. Examples of such graphs are given in [1, 3]. The problem of determining the values of k for which all 2-connected, k-regular graphs on n vertices are hamiltonian was first suggested by G. Szekeres. Erdös and Hobbs [3] proved that such graphs are hamiltonian if $n \leqslant 2k + ck^{1/2}$, where c is a positive constant. Subsequently, Bollobás and Hobbs [1] showed that G is hamiltonian if $n \leqslant \frac{9}{4}k$.

We shall in fact prove a result slightly stronger than Theorem 1.

THEOREM 2. *Let G be a 2-connected graph on n vertices with minimum degree k. Suppose that $n \leqslant 3k$ and*

$$\sum_{v \in V(G)} (d(v) - k) \leqslant k - 1.$$

Then G is hamiltonian.

Thus the regularity condition of Theorem 1 may be relaxed somewhat. The upper bound for $\sum_{v \in V(G)} (d(v)-k)$ cannot be increased since $K_{k+1,k}$ is a non-hamiltonian, 2-connected graph on $2k+1$ vertices, with minimum degree k, and

$$\sum_{v \in V(K_{k+1,k})} (d(v) - k) = k.$$

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0095-8956/80/040027–19$02.00/0

Moreover, the construction described in [1, 3] may be used to construct for all $k \geqslant 3$, a non-hamiltonian, 2-connected graph G on $3k+3$ vertices, with minimum degree k, and

$$\sum_{v \in V(G)} (d(v) - k) = 2.$$

The proof of Theorem 2 is by contradiction. We first prove some general lemmas concerning the distribution of edges in non-hamiltonian graphs. Thus, in the following, let G be a non-hamiltonian graph on n vertices with minimum degree k. Choose a cycle C of maximum length in G so that the number of components of $R = G - C$ is as small as possible. In the first part of the paper we generalize a result of Woodall [7, Theorem 12.3] to deal with the case when R contains an isolated vertex. Nash–Williams [6, Lemma 3] has shown that R consists entirely of isolated vertices if G is 2-connected and $n \leqslant 3k - 2$. In the second part of the paper we adapt his proof to investigate the cases $n = 3k - 1$ and $n = 3k$ in which R contains no isolated vertices.

For H a subgraph of G, let $V(H)$ denote the set of vertices of H. For $v \in V(G)$, let $N_H(v)$ denote the set, and $d_H(v)$ the number, of neighbors of v in H. In order to simplify notation we shall denote $V(G)$, $N_G(v)$, and $d_G(v)$ by V, $N(v)$, and $d(v)$, respectively. For $U \subseteq V$, put

$$N(U) = \bigcup_{v \in U} N(v).$$

If $A, B \subseteq V$, let $\varepsilon(A)$ denote the number of edges in G between the vertices of A, and $\varepsilon(A, B)$ the number of edges joining vertices of A to vertices of B.

Put $r = |V(R)|$ and let $c_1, c_2, \ldots, c_{n-r}, c_1$ be the vertices in order around C. The subscripts of the c_i will be reduced modulo $n - r$ throughout. For $A \subseteq V(C)$, put

$$A^+ = \{c_{i+1} \mid c_i \in A\} \qquad \text{and} \qquad A^- = \{c_{i-1} \mid c_i \in A\}.$$

CASE 1. *R contains an isolated vertex v_0.*

Following Woodall [7] put $Y_0 = \varnothing$ and, for $i \geqslant 1$, put

$$X_i = N(Y_{i-1} \cup \{v_0\})$$

and

$$Y_i = \{c_j \in V(C) \mid c_{j-1} \in X_i \text{ and } c_{j+1} \in X_i\}.$$

Then

$$N(v_0) = X_1 \subseteq X_2 \subset \cdots, \qquad \text{and} \qquad \varnothing = Y_0 \subseteq Y_1 \subseteq \cdots.$$

By [7, Lemma 12.3], $X = \bigcup_{i=1}^{\infty} X_i \subseteq V(C)$, and X does not contain two consecutive vertices of C. Put $Y = \bigcup_{i=1}^{\infty} Y_i$. As consequences of the above definitions and [7, Lemma 12.3]

(i) $Y = X^+ \cap X^-$,

(ii) $N(Y) \subseteq X$, and

(iii) $X \cap Y = \varnothing$.

The following lemma is a slight generalization of a result obtained in the proof of [7, Lemma 12.3].

LEMMA 1. *The following statement* $A(h)$ *holds for all* $h \geqslant 1$.
$A(h)$: *there does not exist a path* $Q_h = q_1 q_2 \cdots q_z$ *in* $G - v_0$ *such that*

(a) $V(C) \subseteq V(Q_h)$,

(b) $q_1, q_z \in X_h$, *and*

(c) *for all ordered pairs* (i, l) *such that* $q_i \in Y_l$ *and* $l \leqslant h-1$, *the vertices* q_{i-1} *and* q_{i+1} *belong to* X_l.

Proof. Suppose that the lemma is false. Let j be the smallest integer for which there exists a path $Q_j = q_1 q_2 \cdots q_z$ which is a counterexample to $A(j)$. If $j = 1$, then adjoining the path $q_1 v_0 q_z$ to q_1 creates a longer cycle than C. This contradicts the choice of C and hence $j \geqslant 2$. Consider the following four cases.

(i) $q_1, q_z \in X_{j-1}$. Clearly Q_j is a counterexample to $A(j-1)$.

(ii) $q_1 \in X_{j-1}$ and $q_z \in X_j \backslash X_{j-1}$. By the definition of X_j, $q_z \in N(q_m)$ for some $q_m \in Y_{j-1} \backslash Y_{j-2}$, and $q_m \neq q_1$ since $X \cap Y = \varnothing$. Hence by (c), $q_{m-1}, q_{m+1} \in X_{j-1}$. Thus, the path

$$Q_j' = q_1 q_2 \cdots q_m q_z q_{z-1} \cdots q_{m+1}$$

is a counterexample to $A(j-1)$ since the only vertices of Q_j' which could fail to satisfy condition (c) of $A(j-1)$ are q_m and q_z, neither of which is in Y_{j-2}.

(iii) $q_1 \in X_j \backslash X_{j-1}$ and $q_z \in X_{j-1}$. Using a construction similar to that in (ii), we may again construct a counterexample to $A(j-1)$.

(iv) $q_1, q_z \in X_j \backslash X_{j-1}$. Then $q_1 \in N(q_w)$ and $q_z \in N(q_m)$ for some $q_w, q_m \in Y_{j-1} \backslash Y_{j-2}$ and again $q_w \neq q_z$ and $q_m \neq q_1$. Hence q_{w-1}, q_{w+1}, q_{m-1}, $q_{m+1} \in X_{j-1}$. If $w \leqslant m$, then

$$Q_j' = q_{w-1} q_{w-2} \cdots q_1 q_w q_{w+1} \cdots q_m q_z q_{z-1} \cdots q_{m+1}$$

is a counterexample to $A(j-1)$, while if $w>m$, then

$$Q_j''=q_{m-1}q_{m-2}\cdots q_1q_wq_{w+1}\cdots q_zq_mq_{m+1}\cdots q_{w-1}$$

is a counterexample to $A(j-1)$ since the only vertices which could fail to satisfy condition (c) of $A(j-1)$ are q_1, q_z, q_w, and q_m, none of which are in Y_{j-2}. In each case we construct a counterexample to $A(j-1)$, which contradicts the minimality of j. ∎

COROLLARY 1. *Let $Z^+=X^+\backslash Y$ and $Z^-=X^-\backslash Y$. Then*

(a) *Z^+ and Z^- are independent sets of vertices,*

(b) *given $c_i\in Z^+$ and $e_j\in Z^-$ there do not exist neighbors b_i of c_i and b_j of c_j which are consecutive on C and lie in the set $\{c_{i-2},c_{i-3},\ldots,c_{j+2}\}$,*

(c) *given $c_i,c_j\in Z^+$ or $c_i,c_j\in Z^-$ there does not exist $c_m\in\{c_{i+2},c_{i+3},\ldots,c_{j-1}\}$ such that c_i is joined to c_m and c_j to c_{m-1}, and*

(d) *no vertex of $R-v_0$ is joined to two vertices of Z^+ or two vertices of Z^-.*

Proof. (a) Let $c_i,c_j\in Z^+$ and suppose c_i is joined to c_j. Then $c_i,c_j\in X_h^+\backslash Y$ for some $h\geqslant 1$, and the path

$$Q_h=c_{i-1}c_{i-2}\cdots c_jc_ic_{i+1}\cdots c_{j-1}$$

contradicts Lemma 1, since the only vertices which could fail to satisfy condition (c) of $A(h)$ are c_i and c_j, neither of which is in Y. A similar proof holds if $c_i,c_j\in Z^-$.

(b) Choose $h\geqslant 1$ such that $c_i\in X_h^+\backslash Y$ and $c_j\in X_h^-\backslash Y$. If $b_i=c_m$ and $b_j=c_{m+1}$ then the path

$$Q_h'=c_{i-1}c_{i-2}\cdots c_{m+1}c_jc_{j-1}\cdots c_ic_mc_{m-1}\cdots c_{j+1}$$

contradicts Lemma 1, since $c_i,c_j\notin Y$ by definition, and $c_m,c_{m+1}\notin Y$ since this would create consecutive vertices c_i, c_{i-1} or c_j, c_{j+1} in X. Similarly, if $b_i=c_m$ and $b_j=c_{m-1}$ then the path

$$Q_h''=c_{i-1}c_{i-2}\cdots c_mc_ic_{i+1}\cdots c_jc_{m-1}c_{m-2}\cdots c_{j+1}$$

contradicts Lemma 1.

(c) Suppose $c_i,c_j\in Z^+$. Then $c_i,c_j\in X_h^+\backslash Y$ for some $h\geqslant 1$, and

$$Q_h=c_{i-1}c_{i-2}\cdots c_jc_{m-1}c_{m-2}\cdots c_ic_mc_{m+1}\cdots c_{j-1}$$

contradicts Lemma 1. A similar proof holds if $c_i,c_j\in Z^-$.

(d) Suppose $u \in V(R - v_0)$ is joined to $c_i, c_j \in Z^+$. Then $c_i, c_j \in X_h^+ \backslash Y$ for some $h \geqslant 1$, and

$$Q_h = c_{i-1}c_{i-2} \cdots c_j u c_i c_{i+1} \cdots c_{j-1}$$

contradicts Lemma 1. A similar proof holds if $c_i, c_j \in Z^-$. ∎

Put $x = |X|$, $y = |Y|$, and let $S_1, S_2, \dots, S_x$ be the sets of vertices contained in the open segments of C between vertices of X. Put $s_i = |S_i|$ and $\mathscr{S} = \{S_i \mid s_i \geqslant 2,\ 1 \leqslant i \leqslant x\}$. Note that

$$|\mathscr{S}| = x - y \qquad \text{and} \qquad \sum_{S_i \in \mathscr{S}} s_i = n - r - x - y. \tag{1}$$

In the following let

$$S_i = \{c_l, c_{l+1}, \dots, c_m\} \qquad \text{and} \qquad S_j = \{c_w, c_{w+1}, \dots, c_z\}$$

be distinct elements of $\mathscr{S}$.

Lemma 2. *$\varepsilon(\{c_w, c_z\}, S_i) \leqslant s_i - 1$. Moreover, equality holds if and only if there exist integers g and h such that*

$$N(c_z) \cap S_i = \{c_l, c_{l+1}, \dots, c_g\} \cup \{c_{g+2}, c_{g+4}, \dots, c_{h-2}\}$$

and

$$N(c_w) \cap S_i = \{c_{g+2}, c_{g+4}, \dots, c_{h-2}\} \cup \{c_h, c_{h+1}, \dots, c_m\},$$

where $l - 1 \leqslant g \leqslant h \leqslant m + 1$, and the set $\{c_a, c_{a+e}, c_{a+2e}, \dots, c_b\}$ is empty if $a > b$.

Proof. Put $A = N(c_w) \cap S_i$ and $B = N(c_z) \cap S_i$. Then $c_l \notin A$ by Corollary 1(a), and hence $|A| \leqslant s_i - 1$. By Corollary 1(a), (b), c_z is not joined to any element of $A^+ \cup A^- \cup \{c_m\}$ and thus

$$B \subseteq S_i \backslash (A^+ \cup A^- \cup \{c_m\}).$$

However,

$$|S_i \cap (A^+ \cup A^- \cup \{c_m\})| \geqslant |A| + 1.$$

Hence,

$$|B| \leqslant s_i - (|A| + 1),$$

and

$$\varepsilon(\{c_w, c_z\}, S_i) = |A| + |B| \leqslant |A| + s_i - (|A| + 1) = s_i - 1.$$

Moreover, equality occurs if and only if

$$|S_i \cap (A^+ \cup A^- \cup \{c_m\})| = |A| + 1 \qquad \text{and} \qquad B = S_i \backslash (A^+ \cup A^- \cup \{c_m\}).$$

However,

$$|S_i \cap (A^- \cup \{c_m\})| = |A| + 1.$$

Thus if equality occurs, then $A^+ \cap S_i \subseteq A^- \cup \{c_m\}$. Hence, if $c_e \in A$ and $e \leqslant m-2$, then $c_{e+2} \in A$. Clearly, this implies that the condition of the lemma holds. ∎

The interval S_i is said to be *ψ-connected* to S_j if s_i is odd and c_w and c_z are both joined to c_{l+e} for all odd e, $1 \leqslant e \leqslant m-l-1$.

COROLLARY 2. *Let $S_i, S_j \in \mathscr{S}$. Then*

$$\varepsilon(S_i, S_j) \leqslant (s_i - 1)(s_j - 1) + 1$$

with equality only if S_i and S_j are ψ-connected to each other.

Proof. Let S_i and S_j be as defined above. Then

$$\begin{aligned}\varepsilon(S_i, S_j) = {} & \varepsilon(S_i \backslash \{c_l, c_m\}, S_j \backslash \{c_w, c_z\}) + \varepsilon(\{c_l, c_m\}, S_j) + \varepsilon(\{c_w, c_z\}, S_i) \\ & - \varepsilon(\{c_l, c_m\}, \{c_w, c_z\}).\end{aligned}$$

Clearly

$$\varepsilon(S_i \backslash \{c_l, c_m\}, S_j \backslash \{c_w, c_z\}) \leqslant (s_i - 2)(s_j - 2).$$

By Lemma 2,

$$\varepsilon(\{c_l, c_m\}, S_j) \leqslant s_j - 1 \qquad \text{and} \qquad \varepsilon(\{c_w, c_z\}, S_i) \leqslant s_i - 1.$$

Moreover, if equality occurs then, by Lemma 2,

$$\varepsilon(\{c_l, c_m\}, \{c_w, c_z\}) = 0$$

if and only if S_i and S_j are ψ-connected to each other. Hence

$$\varepsilon(S_i, S_j) \leqslant (s_i - 2)(s_j - 2) + (s_i - 1) + (s_j - 1) = (s_i - 1)(s_j - 1) + 1,$$

with equality only if S_i and S_j are ψ-connected to each other. ∎

LEMMA 3. *Let S_i and S_j be defined as above. If S_i is ψ-connected to S_j then*

$$\varepsilon(Z^+ \cup Z^-, \{c_l, c_{l+2}, \ldots, c_m\}) = 0.$$

Proof. Choose $c_e \in Z^+ \cup Z^-$ and, without loss of generality, assume that $c_e \in Z^+$. If $c_e \in \{c_m, c_{m+1}, \ldots, c_w\}$ then

$$\varepsilon(\{c_e\}, \{c_l, c_{l+2}, \ldots, c_m\}) = 0$$

by Corollary 1(b) and the hypothesis of the lemma. If $c_e \in \{c_{z+2}, c_{z+3}, \ldots, c_l\}$ then

$$\varepsilon(\{c_e\}, \{c_l, c_{l+2}, \ldots, c_m\}) = 0$$

by Corollary 1(a,c) and the hypothesis of the lemma. ∎

If S_i and S_j are as defined above and S_i is ψ-connected to S_j, define c_{l+1}, $c_{l+3}, \ldots, c_{m-1}$ to be the *popular* vertices of S_i and $c_l, c_{l+2}, \ldots, c_m$ to be the *unpopular* vertices of S_i. Let

$$\begin{aligned} \mathscr{S}^* &= \{S_i \in \mathscr{S} \mid S_i \text{ is } \psi\text{-connected to some } S_j \in \mathscr{S}\}, \\ P &= \{c_i \in V(C) \mid c_i \text{ is a popular vertex of some } S_j \in \mathscr{S}^*\}, \\ U &= \{c_i \in V(C) \mid c_i \text{ is an unpopular vertex of some } S_j \in \mathscr{S}^*\}. \end{aligned}$$

Further, let $P_i = P \cap S_i$ and $U_i = U \cap S_i$. Put $s^* = |\mathscr{S}^*|$, $p = |P|$, $p_i = |P_i|$, and $u_i = |U_i|$. Then

$$u_i = p_i + 1 \qquad \text{and} \qquad s_i = 2p_i + 1. \tag{2}$$

LEMMA 4. *Let S_i and S_j be as defined above.*

(a) *If $S_i, S_j \in \mathscr{S} \backslash \mathscr{S}^*$ then*

$$\varepsilon(S_i) \leqslant \tfrac{1}{2} s_i(s_i - 1)$$

and

$$\varepsilon(S_i, S_j) \leqslant (s_i - 1)(s_j - 1).$$

(b) *If $S_i \in \mathscr{S} \backslash \mathscr{S}^*$ and $S_j \in \mathscr{S}^*$ then*

$$\varepsilon(S_i, U_j) \leqslant (s_i - 2) p_j.$$

(c) *If $S_i, S_j \in \mathscr{S}^*$ then*

$$\varepsilon(U_j) \leqslant \tfrac{1}{2}(p_j - 1)(p_j - 2)$$

and

$$\varepsilon(U_i, U_j) \leqslant (p_i - 1)(p_j - 1).$$

Proof. (a) This follows immediately from Corollary 2.

(b) Clearly

$$\varepsilon(S_i\backslash\{c_l, c_m\}, U_j\backslash\{c_w, c_z\}) \leqslant (s_i - 2)(u_j - 2).$$

By Lemma 3,

$$\varepsilon(\{c_l, c_m\}, \{c_w, c_z\}) = 0.$$

Thus, the condition for equality in Lemma 2 could occur if and only if S_i was ψ-connected to S_j, which would contradict the assumption that $S_i \in \mathscr{S}\backslash\mathscr{S}^*$. Hence

$$\varepsilon(S_i, \{c_w, c_z\}) \leqslant s_i - 2,$$

and, by (2),

$$\varepsilon(S_i, U_j) \leqslant (s_i - 2)(u_j - 2) + (s_i - 2) = (s_i - 2)p_j.$$

(c) By Lemma 3, $\varepsilon(\{c_w, c_z\}, U_j) = 0$. Hence

$$\varepsilon(U_j) \leqslant \tfrac{1}{2}(u_j - 2)(u_j - 3) = \tfrac{1}{2}(p_j - 1)(p_j - 2).$$

Again, using Lemma 3,

$$\varepsilon(\{c_l, c_m\}, U_j) = 0 = \varepsilon(\{c_w, c_z\}, U_i).$$

Hence

$$\varepsilon(U_i, U_j) \leqslant (u_i - 2)(u_j - 2) = (p_i - 1)(p_j - 1). \quad \blacksquare$$

LEMMA 5.

$$\varepsilon(V(C)\backslash(X \cup P)) \leqslant \tfrac{1}{2}(n - r - 2x - p)(n - r - 2x - p + 1) - p(x - y + 1) + \tfrac{1}{2}s^*(s^* + 1).$$

Proof. For convenience, we shall write $\sum_i$ and $\sum_i^*$ to denote $\sum_{S_i \in \mathscr{S}\backslash\mathscr{S}^*}$ and $\sum_{S_i \in \mathscr{S}^*}$, respectively. In an obvious notation, we have:

$$\sum_i^* \sum_{j \neq i}^* p_j = \sum_i^* \sum_{j \neq i}^* p_i, \tag{3}$$

$$\sum_i 1 = |\mathscr{S}\backslash\mathscr{S}^*| = x - y - s^* \qquad \text{(by (1))}, \tag{4}$$

$$\sum_i^* 1 = |\mathscr{S}^*| = s^*, \tag{5}$$

$$\sum_i^* p_i = p, \tag{6}$$

and

$$\sum_i (s_i - 1) + \sum_j^* p_j = \sum_{S_i \in \mathscr{S}} (s_i - 1) - \sum_j^* p_j \qquad \text{(by (2))}$$
$$= (n - r - x - y) - (x - y) - p$$
$$\text{(by (1) and (6))}$$
$$= n - r - 2x - p. \tag{7}$$

Now

$$V(C) \backslash (X \cup P) = \Big(\bigcup_{S_i \in \mathscr{S} \backslash \mathscr{S}^*} S_i \Big) \cup \Big(\bigcup_{S_j \in \mathscr{S}^*} U_j \Big) \cup Y.$$

Since $N(Y) \subseteq X$,

$$\varepsilon(V(C) \backslash (X \cup P))$$
$$= \varepsilon \Big(\bigcup_{S_i \in \mathscr{S} \backslash \mathscr{S}^*} S_i \Big) + \varepsilon \Big(\bigcup_{S_i \in \mathscr{S} \backslash \mathscr{S}^*} S_i, \bigcup_{S_j \in \mathscr{S}^*} U_j \Big) + \varepsilon \Big(\bigcup_{S_j \in \mathscr{S}^*} U_j \Big).$$

Using Lemma 4 we see that

$$\varepsilon(V(C) \backslash (X \cup P))$$
$$\leqslant \Big[\sum_i \frac{1}{2} s_i(s_i - 1) + \frac{1}{2} \sum_i \sum_{j \neq i} (s_i - 1)(s_j - 1) \Big] + \Big[\sum_i \sum_j^* (s_i - 2) p_j \Big]$$
$$+ \Big[\sum_j^* \frac{1}{2} (p_j - 1)(p_j - 2) + \frac{1}{2} \sum_i^* \sum_{j \neq i}^* (p_i - 1)(p_j - 1) \Big]$$
$$= \frac{1}{2} \Big[\Big(\sum_i (s_i - 1) \Big)^2 + \sum_i (s_i - 1) \Big] + \frac{1}{2} \Big[\sum_i \sum_j^* (2(s_i - 1) p_j - 2p_j) \Big]$$
$$+ \frac{1}{2} \Big[\Big(\sum_j^* p_j \Big)^2 + \sum_j^* p_j - \sum_j^* (4p_j - 2)$$
$$- \sum_i^* \sum_{j \neq i}^* (2p_j - 1) \Big] \qquad \text{(by (3))}$$
$$= \frac{1}{2} \Big[\Big(\sum_i (s_i - 1) + \sum_j^* p_j \Big)^2 + \Big(\sum_i (s_i - 1) + \sum_j^* p_j \Big)$$
$$- 2(x - y - s^*) \sum_j^* p_j - (s^* + 1) \sum_j^* (2p_j - 1) \Big] \qquad \text{by (4) and (5))}$$

$$= \frac{1}{2} [(n-r-2x-p)^2 + (n-r-2x-p) - 2p(x-y+1)$$
$$+ s^*(s^*+1)] \qquad \text{(by (7), (6), and (5))}$$
$$= \frac{1}{2} (n-r-2x-p)(n-r-2x-p+1) - p(x-y+1) + \frac{1}{2} s^*(s^*+1)$$

as required. ∎

COROLLARY 5.

$$2\varepsilon(V(C)\backslash(X \cup P)) \leqslant (n-r-2x-p)(n-r-2x-p+1) - p(x-y+1). \quad (8)$$

Proof. By the definition of $\mathscr{S}^*$, $s^* \leqslant p$ and $s^* \leqslant |\mathscr{S}| = x-y$. Thus $s^*(s^*+1) \leqslant p(x-y+1)$ and the corollary follows immediately from Lemma 5.

Proof of Theorem 2 *for the Case when R Contains an Isolated Vertex* v_0

Let G be as above and suppose further that G satisfies the hypotheses of Theorem 2. Then

$$\phi = \sum_{v \in V} (d(v) - k) \leqslant k - 1. \quad (9)$$

By considering two bounds on $\varepsilon(V\backslash X)$ we shall show that $n > 3k$, and thus obtain a contradiction. Put

$$\xi = \sum_{v \in X} (d(v) - k).$$

Clearly $\varepsilon(X, V\backslash X) \leqslant xk + \xi$. However,

$$\varepsilon(V\backslash X, X) = (n-x)k + (\phi - \xi) - 2\varepsilon(V\backslash X).$$

Hence

$$2\varepsilon(V\backslash X) \geqslant (n-2x)k + \phi - 2\xi. \quad (10)$$

On the other hand, $V\backslash X = P \cup (V(C)\backslash(X \cup P)) \cup V(R)$. Thus

$$\varepsilon(V\backslash X) = \varepsilon(P, V\backslash X) + \varepsilon(V(C)\backslash(X \cup P)) + \varepsilon(V(R)) + \varepsilon(V(R), V(C)\backslash(X \cup P)). \quad (11)$$

Since R contains at least one isolated vertex,

$$\varepsilon(V(R)) \leqslant \tfrac{1}{2}(r-1)(r-2). \quad (12)$$

Put $F = V(C)\backslash(X \cup X^+ \cup X^-)$ and $f = |F|$. Then

$$f = n - r - x - 2x + y, \tag{13}$$

and

$$\varepsilon(V(R), V(C)\backslash(X \cup P)) = \varepsilon(V(R), F\backslash P) + \varepsilon(V(R), X^+ \cup X^-).$$

Clearly $\varepsilon(V(R), F\backslash P) \leqslant (r-1)(f-p)$, since $\varepsilon(\{v_0\}, F\backslash P) = 0$. By Corollary 1(d), $\varepsilon(V(R), X^+ \cup X^-) \leqslant 2(r-1)$. Thus

$$\varepsilon(V(R), V(C)\backslash(X \cup P) \leqslant (r-1)(f-p+2). \tag{14}$$

Substituting (8), (12), and (14) into (11) gives

$$\begin{aligned}2\varepsilon(V\backslash X) \leqslant{}& 2\varepsilon(P, V\backslash X) + (n-2x-r-p)(n-2x-r-p+1) - p(x-y+1)\\ &+ (r-1)(r-2) + 2(r-1)(f-p+2).\end{aligned}$$

Putting $\bar{r} = r - 1$, we see that

$$\begin{aligned}&(n-2x-r-p)(n-2x-r-p+1)\\ &\quad = (n-2x-2p-1)(n-2x) + p(p+1) - \bar{r}(2n-4x-\bar{r}-2p-1).\end{aligned}$$

Thus

$$\begin{aligned}2\varepsilon(V\backslash X) &\leqslant 2\varepsilon(P, V\backslash X) + (n-2x-2p-1)(n-2x) + p(p+1)\\ &\quad - \bar{r}(2n-4x-\bar{r}-2p-1) - p(x-y+1) + \bar{r}(2f-2p+\bar{r}+3)\\ &= 2\varepsilon(P, V\backslash X) + (n-2x-2p-1)(n-2x) - p(x-y-p)\\ &\quad - \bar{r}(2n-4x-2\bar{r}-2f-4).\end{aligned}$$

Put $\theta = \bar{r}(2n-4x-2\bar{r}-2f-4)$. From (13), $f = n-2x-(x-y)-\bar{r}-1$, and hence

$$\theta = 2\bar{r}(x-y-1).$$

Suppose $x-y-1 < 0$. By the definition of X and Y, $x \geqslant y$. Thus $x = y$ and

$$\varepsilon(Y \cup \{v_0\}, X) \geqslant (y+1)k = xk + k.$$

However, this contradicts the fact that

$$\varepsilon(X, Y \cup \{v_0\}) \leqslant xk + \phi \leqslant xk + k - 1$$

by (9). Thus we may assume that $x-y-1 \geqslant 0$. Hence $\theta \geqslant 0$, and

$$2\varepsilon(V\backslash X) \leqslant 2\varepsilon(P, V\backslash X) + (n-2x-2p-1)(n-2x) - p(x-y-p). \tag{15}$$

Clearly $V\backslash X = Y \cup \{v_0\} \cup P \cup (V(R)\backslash\{v_0\}) \cup (V(C)\backslash(X \cup Y \cup P))$. Put $M = (V(R)\backslash\{v_0\}) \cup (V(C)\backslash(X \cup Y \cup P))$. Then

$$|M| = n - x - y - p - 1.$$

Since $\varepsilon(P, Y) = \varepsilon(P, \{v_0\}) = 0$, it follows that

$$\varepsilon(P, V\backslash X) = \varepsilon(P) + \varepsilon(P, M).$$

We shall now prove that

$$\varepsilon(P, V\backslash X) \leqslant \tfrac{1}{2}p(k + n - x - y - p - 1) + \phi - \xi. \tag{16}$$

If $(n - x - y - p - 1) + (p - 1) \leqslant k$, then $\varepsilon(P, V\backslash X)$ is largest when each vertex of P is joined to every vertex of P and every vertex of M. Thus

$$\begin{aligned}\varepsilon(P, V\backslash X) &\leqslant \tfrac{1}{2}p(p-1) + p(n - x - y - p - 1)\\ &\leqslant \tfrac{1}{2}p(p-1) + \tfrac{1}{2}p(n - x - y - p - 1) + \tfrac{1}{2}p(k - p + 1)\\ &\leqslant \tfrac{1}{2}p(k + n - x - y - p - 1) + \phi - \xi.\end{aligned}$$

If $n - x - y - p - 1 \leqslant k < (n - x - y - p - 1) + (p - 1)$, then $\varepsilon(P, V\backslash X)$ is largest when each vertex of P is joined to every vertex of M and to $k - |M|$ vertices of P, and there are an additional $\tfrac{1}{2}(\phi - \xi)$ edges between the vertices of P. Thus

$$\begin{aligned}\varepsilon(P, V\backslash X) &\leqslant p(n - x - y - p - 1) + \tfrac{1}{2}p(k - (n - x - y - p - 1)) + \tfrac{1}{2}(\phi - \xi)\\ &\leqslant \tfrac{1}{2}p(k + n - x - y - p - 1) + \phi - \xi.\end{aligned}$$

Finally, if $k < n - x - y - p - 1$, then clearly

$$\begin{aligned}\varepsilon(P, V\backslash X) &\leqslant pk + \phi - \xi\\ &\leqslant \tfrac{1}{2}pk + \tfrac{1}{2}p(n - x - y - p - 1) + \phi - \xi\\ &= \tfrac{1}{2}p(k + n - x - y - p - 1) + \phi - \xi.\end{aligned}$$

In each case (16) holds. Combining (16) with (15) yields

$$\begin{aligned}2\varepsilon(V\backslash X) &\leqslant p(k + n - x - y - p - 1) + 2(\phi - \xi) + (n - 2x - 2p - 1)(n - 2x)\\ &\quad - p(x - y - p)\\ &= (n - 2x - 1)(n - 2x) - p(n - 2x + 1 - k) + 2(\phi - \xi).\end{aligned}$$

Thus, from (10),

$$(n - 2x)k + \phi - 2\xi \leqslant (n - 2x - 1)(n - 2x) - p(n - 2x + 1 - k) + 2\phi - 2\xi$$

and, by (9),

$$\begin{aligned}(n-2x)k &\leqslant (n-2x-1)(n-2x)-p(n-2x+1-k)+\phi\\ &\leqslant (n-2x-1)(n-2x)-p(n-2x+1-k)+k-1.\end{aligned}$$

Hence

$$\begin{aligned}0 &\leqslant (n-2x-1)(n-2x)-(n-2x-1)k-p(n-2x+1-k)-1\\ &< (n-2x-1-p)(n-2x-k).\end{aligned}$$

By the definition of P, $p \leqslant \frac{1}{2}(n-1-2x)$. Thus,

$$n-2x-1-p \geqslant 2p-p=p \geqslant 0.$$

Hence $n-2x-k>0$. However, $x \geqslant k$, since $N(v_0) \subseteq X$, and so $n > 2x+k \geqslant 3k$. This contradicts the original hypothesis of the theorem, and completes the discussion of Case 1.

CASE 2. *R contains no isolated vertices.*

Let G be as defined at the beginning of the paper. Given a path Q in R with end points q_1 and q_g, let $t(Q)$ be the number of occurrences of ordered pairs, (c_i, c_j), of the vertices of C such that c_i is joined to one of q_1 and q_g, c_j is joined to the other, and

$$\varepsilon(\{q_1, q_g\}, \{c_{i+1}, c_{i+2}, \ldots, c_{j-1}\})=0.$$

LEMMA 6. *Let* $Q=q_1q_2\cdots q_g$ *be a maximal path in* R *such that* $t(Q) \geqslant 2$. *Put* $h=t(Q)-2$ *and* $w=r-g$. *Then*

(a) $n \geqslant 3k-1+h(g-2)+w$, *and*

(b) *if* $n \leqslant 3k$, *then* $N_C(q_1)=N_C(q_g)$.

Proof. Put $A=N_C(q_1)$, $B=N_C(q_g)$, and $t=t(Q)$. Without loss of generality we can assume $|A| \leqslant |B|$. If $c_i, c_j \in A$ (or $c_i, c_j \in B$) then $|\{c_{i+1}, c_{i+2}, \ldots, c_{j-1}\}| \geqslant 1$ for, otherwise there would exist a longer cycle than C. Similarly, if $c_i \in A$ and $c_j \in B$ (or $c_i \in B$ and $c_j \in A$) then $|\{c_{i+1}, c_{i+2}, \ldots, c_{j-1}\}| \geqslant g$. Hence

$$\begin{aligned}|V(C)|=n-r &\geqslant |A \cup B|+|A^+ \cup A^- \cup B^+ \cup B^-|+t(g-2)\\ &=|A|+|A^+ \cup B^-|+|B \backslash A|+|(A^- \cup B^+)\backslash(A^+ \cup B^-)|+t(g-2).\end{aligned}$$

Since $g \geqslant 2$, $A^+ \cap B^- = \varnothing = A^- \cap B^+$. Putting

$$\sigma=|B \backslash A|+|(A^- \cup B^+)\backslash(A^+ \cup B^-)|$$

we see that

$$n - r \geqslant |A| + |A^+| + |B^-| + t(g-2) + \sigma.$$

Since Q is a maximal path, $d_R(q_1) \leqslant g - 1$, and thus

$$|A| = d_C(q_1) \geqslant k - (g-1).$$

Similarly,

$$|B| \geqslant k - (g-1).$$

Furthermore, $|A| = |A^+| = |A^-|$ and $|B| = |B^+| = |B^-|$. Hence

$$n - r \geqslant 3k - 3(g-1) + t(g-2) + \sigma$$

and

$$\begin{aligned} n &\geqslant 3k - 3(g-1) + (h+2)(g-2) + \sigma + g + w \\ &= 3k - 1 + h(g-2) + w + \sigma, \end{aligned}$$

which proves part (a) since $\sigma \geqslant 0$. Suppose $A \neq B$. Then $|B \backslash A| \geqslant 1$, by the assumption that $|A| \leqslant |B|$. Furthermore, we can choose $c_i \in B \backslash A$ such that if c_j is the next element of $A \cup B$ around C after c_i, then $c_j \in A$. Clearly

$$c_{i+1} \in (A^- \cup B^+) \backslash (A^+ \cup B^-)$$

and

$$|(A^- \cup B^+) \backslash (A^+ \cup B^-)| \geqslant 1.$$

Thus $\sigma \geqslant 2$ and $n \geqslant 3k + 1 + h(g-2) + w$. Hence if $n \geqslant 3k$, we must have $A = B$, which completes the proof of (b). ∎

In the following we suppose, further, that G is 2-connected and $n \leqslant 3k$. Under these conditions we show, using a result of Dirac, that G contains a path Q which satisfies the hypotheses of Lemma 6.

LEMMA 7 [2, Theorem 4]. *If H is a 2-connected graph on n vertices with minimum degree k, then H contains a cycle of length at least* $\min(2k, n)$.

COROLLARY 7. *Let $u \in V(R)$. Then $d_C(u) \geqslant 1$.*

Proof. The cycle C is of maximum length in G and thus, by Lemma 7, contains at least $2k$ vertices. Since $n \leqslant 3k$,

$$|V(R)| \leqslant n - 2k \leqslant k \tag{17}$$

and

$$d_R(u) \leqslant k-1.$$

Since $d_C(u)+d_R(u)=d(u) \geqslant k$, it follows that

$$d_C(u) \geqslant 1. \quad \blacksquare$$

LEMMA 8. *If R contains a component W which is not a single vertex then there exists a maximal path Q in W such that $t(Q) \geqslant 2$.*

Proof. Since G is 2-connected there exist two distinct vertices, a and b, of W such that a is joined to c_i and b to c_j, where c_i and c_j are distinct vertices of C. Furthermore, a and b are connected by a path Q in W. Let a, b, and Q be chosen so that Q is as long as possible and suppose $Q=q_1q_2 \cdots q_g$, where $q_1=a$ and $q_g=b$. We shall show that Q is a maximal path in W and hence, since $t(Q) \geqslant 2$, satisfies the conditions of the lemma. The proof is by contradiction. Suppose Q is not a maximal path. Then, without loss of generality, we can assume Q can be extended to a path

$$Q'=q_1q_2 \cdots q_gq_{g+1} \cdots q_e,$$

where q_e is not joined to any vertex of $W-Q'$. Moreover, q_e can be joined to at most one vertex, c_i, of C since Q' is longer than Q and, if q_e were joined to $c_l \in V(C)$, $c_l \neq c_i$, this would contradict the choice of Q. Using (17) and the fact that $d(q_e) \geqslant k$, we see that

$$d_R(q_e) \geqslant k-1 \geqslant r-1.$$

Thus q_e is joined to every other vertex of R and the path

$$Q''=q_1q_2 \cdots q_{g-1}q_eq_{e-1} \cdots q_g$$

contradicts the choice of Q. Hence the assumption that Q is not a maximal path in W is false. $\blacksquare$

COROLLARY 8. *If $n \leqslant 3k-2$, then R is composed of r isolated vertices.*

Proof. Immediate from Lemmas 6(a) and 8. $\blacksquare$

Proof of Theorem 2 for the Case when R Contains No Isolated Vertices.

Let G be as above and suppose, further, that G satisfies the hypotheses of Theorem 2. We may assume that $k \geqslant 3$, since if $k=1$, G is not 2-connected, and if $k=2$, G is a 2-connected, 2-regular graph and thus is hamiltonian.

By Lemma 8, there exists a maximal path $Q=q_1q_2 \cdots q_g$ in R such that $t(Q) \geqslant 2$. Choose Q to be of maximum length with this property. By Lemma 6(a), the following is a complete list of all possible values of h, g, and w: $h=0$ and $w=0$ or 1; $g=2$ and $w=0$ or 1; and $h=1$, $g=3$, and $w=0$. We shall show that none of these sets of values can occur.

(i) Suppose $h=0$ and $w=0$ or 1. Then $t(Q)=2$ and thus, by Lemma 6(b)

$$N_C(q_1)=N_C(q_g)=\{c_i, c_j\}$$

for some $c_i, c_j \in V(C)$. Hence $d_C(q_1)=2$ and

$$d_Q(q_1)=d_R(q_1)\geqslant k-2. \tag{18}$$

Thus

$$|V(Q)|=g\geqslant k-1 \tag{19}$$

and

$$|V(C)|=n-r\leqslant n-g\leqslant 2k+1.$$

Put

$$S_1=\{c_{i+1}, c_{i+2},..., c_{j-1}\} \quad \text{and} \quad S_2=\{c_{j+1}, c_{j+2},..., c_{i-1}\}.$$

Since

$$|S_1|+|S_2|=|V(C)|-2\leqslant 2k-1, \tag{20}$$

we can assume, without loss of generality, that

$$|S_1|\leqslant k-1. \tag{21}$$

Let $C'=q_1 q_2 \cdots q_g c_j c_{j+1} \cdots c_i q_1$. Then

$$|V(C')|=|V(C)|-|S_1|+g.$$

Since C is a longest cycle of G, $|S_1|\geqslant g$, and thus, by (19) and (21),

$$|S_1|=k-1=g. \tag{22}$$

Similarly, one can show that $|S_2|\geqslant g$. Thus, by (20) and (22),

$$k-1\leqslant |S_2|\leqslant k. \tag{23}$$

Using (18) and (22), we see that q_1 is joined to every other vertex of Q.

Suppose $w=1$. Then there exists a vertex $q_0 \in V(R-Q)$. Since R

contains no isolated vertices, q_0 is joined to some vertex $q_l \in V(Q)$. Clearly $q_l \neq q_1$ or q_g since Q is a maximal path. Consider the path

$$Q' = q_0 q_l q_{l-1} \cdots q_1 q_{l+1} q_{l+2} \cdots q_g.$$

By Corollary 7, $d_C(q_0) \geqslant 1$ and hence $t(Q') \geqslant 2$, which contradicts the choice of Q as a longest path such that $t(Q) \geqslant 2$. Thus we may assume that $w = 0$, and hence $R = Q$. Consider the $k-2$ paths

$$Q_m = q_m q_{m-1} \cdots q_1 q_{m+1} q_{m+2} \cdots q_g$$

for $1 \leqslant m \leqslant g-1 = k-2$. Each Q_m is a maximal path and $t(Q_m) \geqslant 2$, since $d_C(q_m) \geqslant 1$. By Lemma 6(b),

$$N_C(q_m) = N_C(q_g) = \{c_i, c_j\}, \tag{24}$$

for all m, $1 \leqslant m \leqslant g-1$. Again consider the cycle

$$C' = q_1 q_2 \cdots q_g c_j c_{j+1} \cdots c_i q_1.$$

By (22), C' is a longest cycle in G. Moreover, $G - C'$ is a path $Q' = c_{i+1} c_{i+2} \cdots c_{j-1}$ of length $|S_1| = k-1$. By a similar argument to the above

$$N_{C'}(c_m) = \{c_i, c_j\} \tag{25}$$

for all $c_m \in S_1$. By (24) and (25),

$$\varepsilon(S_2, S_1 \cup V(Q)) = 0. \tag{26}$$

Choose $c_l \in S_2$. By (23)

$$\varepsilon(c_l, S_2) \leqslant |S_2| - 1 \leqslant k-1.$$

Thus, using (26)

$$\varepsilon(c_l, \{c_i, c_j\}) \geqslant 1$$

and

$$\begin{aligned}\varepsilon(S_1 \cup S_2 \cup V(Q), \{c_i, c_j\}) &\geqslant 2\,|S_1| + |S_2| + 2\,|V(Q)| \\ &\geqslant 2(k-1) + (k-1) + 2(k-1) = 5(k-1),\end{aligned}$$

by (22) and (23). However,

$$\varepsilon(\{c_i, c_j\}, S_1 \cup S_2 \cup V(Q)) \leqslant 2k + k - 1 = 3k - 1.$$

This gives a contradiction, since $k \geqslant 3$, and completes the proof of subcase (i).

(ii) Suppose $g = 2$ and $w = 0$ or 1, or $h = 1$, $g = 3$, and $w = 0$. The case $w = 1$ cannot occur since, if there exists a single vertex in $R - Q$, either it is an isolated vertex of R, or the path $Q = q_1q_2$ is not maximal in R. Thus assume that $w = 0$, i.e., $r = g = 2$ or 3. Let $t = t(Q)$, $T = N_C(q_1)$ $(= N_C(q_g))$, and $S_1, S_2, \ldots, S_t$ be the sets of vertices contained in the open segments of C between the vertices of T. Since $|S_i| \geqslant g$ for all i, $1 \leqslant i \leqslant t$,

$$n = \sum_{i=1}^{t} |S_i| + t + g \geqslant tg + t + g. \tag{27}$$

If $g = 2$, then $t = d_C(q_1) \geqslant k - d_R(q_1) = k - 1$. Using (27) and the hypothesis that $n \leqslant 3k$ we may deduce that $t = k - 1$.

If $g = 3$, then $t = h + 2 = 3$. Moreover, $d_R(q_1) \leqslant 2$, and hence $k \leqslant t + d_R(q_1) \leqslant 5$. By (27), $n \geqslant 15$ and since $n \leqslant 3k$, we have $k = 5$ and $n = 15$.

Defining t_j to be the number of sets S_i of cardinality j, it can be seen that $t_2 + t_3 = t$, and that the only three possibilities are those given in the following table.

g	n	t	t_2	t_3
2	$3k - 1$	$k - 1$	$k - 1$	0
2	$3k$	$k - 1$	$k - 2$	1
3	15	3	0	3

Clearly

$$\begin{aligned} \varepsilon(S_i) &= 1 \quad \text{if } |S_i| = 2 \\ &\leqslant 3 \quad \text{if } |S_i| = 3 \end{aligned}$$

Moreover, since C is a longest cycle in G,

$$\begin{aligned} \varepsilon(S_i, S_j) &= 0 \quad \text{if } |S_i| = |S_j| = 2, S_i \neq S_j \\ &\leqslant 1 \quad \text{if } |S_i| = 2 \text{ and } |S_j| = 3. \end{aligned}$$

Hence

$$\varepsilon(V(C) \backslash T) \leqslant t_2 + 3t_3 + t_2t_3$$

and, in each case,

$$\varepsilon(V(C) \backslash T, R) = 0.$$

Thus

$$\begin{aligned}\varepsilon(V(C)\backslash T, T) &\geqslant |V(C)\backslash T|\,k - 2\varepsilon(V(C)\backslash T)\\ &\geqslant (2t_2+3t_3)k - 2(t_2+3t_3+t_2t_3)\\ &= t(k-2) + t_2(k-t_3) + t_3(2k-t_2-4).\end{aligned}$$

It is easily seen that $t_2(k-t_3)+t_3(2k-t_2-4) \geqslant k$ for all possible sets of values of t_2 and t_3. Thus

$$\varepsilon(V(C)\backslash T, T) \geqslant t(k-2)+k.$$

However, since each vertex of T is joined to q_1 and q_g,

$$\varepsilon(T, V(C)\backslash T) \leqslant t(k-2)+k-1.$$

This final contradiction completes the proof of Theorem 2. ∎

For $k \geqslant 4$, it seems likely that all 2-connected, k-regular graphs on at most $3k+3$ vertices are hamiltonian. We feel confident that the proof of Theorem 2 could be extended to show that, with the exception of the Petersen graph, all 2-connected, k-regular graphs on at most $3k+1$ vertices are hamiltonian. The proof of this slightly improved bound would be rather tedious, however, and we feel that more radical changes would be required to obtain the bound $3k+3$.

Other open problems which are related to Theorem 1 are the following conjectures due to Häggkvist.

CONJECTURE 1 [4]. *For $k \geqslant 4$, all m-connected, k-regular graphs on at most $(m+1)k$ vertices are hamiltonian.*

CONJECTURE 2 [5]. *All m-connected, k-regular, bipartite graphs on at most $2(m+1)k$ vertices are hamiltonian.*

Constructions similar to those given in [1, 3] show that these conjectures would again be essentially best possible.

ACKNOWLEDGMENTS

I would like to thank Professor J. A. Bondy for many helpful discussions on this problem, and Professor D. R. Woodall for his suggestions which simplified several of the proofs in the original version of this paper. I also gratefully acknowledge the support of the Canadian Commonwealth Association.

Note added in Proof. Using a construction similar to that for the Meredith graphs, we have recently shown that Conjectures 1 and 2 are false for large values of m by constructing non-hamiltonian $\frac{2}{3}k$-connected, k-regular, (bipartite) graphs on approximately $10k$ $(16k)$ vertices.

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